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Klein–Gordon and Dirac Equations in de Sitter Space–Time

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We present and discuss the Klein–Gordon and Dirac wave equations in the de Sitter universe. To obtain the Dirac wave equation we use the factorization of the second-order invariant Casimir operator associated to the Fantappié–de Sitter group. Both the Klein–Gordon and Dirac wave equations are discussed in terms of the spherical harmonics with spin weight. A particular case of Dirac wave equation is solved in terms of a new class of polynomials.

1. INTRODUCTION

In a recent paper Notte Cuello and Capelas de Oliveira [1] presented in a systematic way a construction of the Casimir invariant operators associated to the Fantappié–de Sitter group. More recently these authors presented a study of the Klein–Gordon wave equation [2] in the de Sitter universe using the so-called Fantappié–Arcidiacono [3] method and discussed the equation in terms of the associated Legendre function. Capelas de Oliveira [4] discussed the homogeneous d'Alembert generalized wave equation for the case of a physical situation involving a small distance (a local problem) using the same technique. More recently [5] the same authors presented and discussed the Dirac wave equation by means of the factorization of the second-order invariant Casimir operator.

Here we discuss Klein–Gordon and Dirac wave equations using spherical harmonics with spin weight introduced by Newman and Penrose [6] and more recently by Torres del Castillo [7], and solve a particular case of Dirac's

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wave equation in terms of the $E_n^l(\rho)$ recently introduced by Gomes and Capelas de Oliveira [8, 9].

This paper is organized as follows: In Section 2 we present and discuss the generalized Klein–Gordon wave equation, in Section 3 we present and discuss the generalized Dirac wave equation and solve this equation for a particular case, and in Section 4 we present our comments.

2. KLEIN–GORDON WAVE EQUATION

In this section we present and discuss the Klein–Gordon wave equation in the de Sitter universe in spherical coordinates and obtain its solution by means of the spin-weight technique.

The Klein–Gordon wave equation is given by [1]

$$I_2\Psi(r, \theta, \phi, t) = -M^2\Psi(r, \theta, \phi, t) \tag{2.1}$$

where M^2 is a constant associated with the mass of the particle and the I_2 operator is given by

$$I_2 = \hbar^2 A^2 \left\{ \left(1 + \frac{r^2}{R^2}\right)\frac{\partial^2}{\partial r^2} + \frac{2rt}{R^2}\frac{\partial^2}{\partial r \partial t} - \left(1 - \frac{t^2c^2}{R^2}\right)\frac{\partial^2}{\partial (ct)^2} + \frac{2}{r}\left(1 + \frac{r^2}{R^2}\right)\frac{\partial}{\partial r} + \frac{2t}{R^2}\frac{\partial}{\partial t} + \frac{1}{r^2}\mathcal{L}^2 \right\} \tag{2.2}$$

where the $\mathcal{L}^2$ operator, given by

$$\mathcal{L}^2 \equiv \frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \phi^2} \tag{2.3}$$

is the usual angular momentum operator.

Introducing the I_2 operator in equation (2.1), we obtain the following partial differential equation:

$$\left\{\nabla^2 + \left[\xi^2\frac{\partial^2}{\partial \xi^2} + 2\xi\eta\frac{\partial^2}{\partial \xi \partial \eta} - (1 - \eta^2)\frac{\partial^2}{\partial \eta^2} + 2\xi\frac{\partial}{\partial \xi} + 2\eta\frac{\partial}{\partial \eta} - \alpha\right]\right\}\Psi = 0 \tag{2.4}$$

where $\Psi \equiv \Psi(\xi, \eta, \theta, \phi)$ and we have put

$$\xi = \frac{r}{R}; \qquad \eta = \frac{ct}{R}; \qquad M_0 = \frac{iRM}{\hbar}; \qquad \alpha = \frac{M_0^2}{A_0^2}$$

$A_0^2 = 1 + \xi^2 - \eta^2$ and ∇^2 is the Laplacian operator in spherical coordinates.

Following ref. 7, we have an arbitrary vector field $\mathcal{F}$ which can be expressed in terms of its components by means of

$$\mathcal{F} = F_r \hat{e}_r + \frac{1}{2} F_-(\hat{e}_\theta + i\hat{e}_\phi) + \frac{1}{2} F_+(\hat{e}_\theta - i\hat{e}_\phi)$$

where F_r, F_+, and F_- have spin weight zero, one, and minus one, respectively, and $\hat{e}_r$, $\hat{e}_\theta$, and $\hat{e}_\phi$ are the orthogonal vectors tangent to the spherical coordinate lines.

Then, using the fact that the set of spherical harmonics with spin weight is a complete set [7], we can suppose for the above equation the following solutions:

$$\begin{aligned} \Psi_\xi &= [l(l+1)]^{1/2} f(\xi, \eta) Y_{lm}(\theta, \phi) \\ \Psi_+ &= g_1(\xi, \eta)\,{}_1Y_{lm}(\theta, \phi) \\ \Psi_- &= g_2(\xi, \eta)\,{}_{-1}Y_{lm}(\theta, \phi) \end{aligned} \tag{2.5}$$

The factor $[l(l+1)]^{1/2}$ is introduced for convenience; the variable η is taken as a constant and we use the fact that the components Ψ_ξ, Ψ_+, and Ψ_- have spin zero, one, and minus one, respectively.

Then, introducing the functions given by (2.5) in equation (2.4) and using the fact that the set $\{\hat{e}_r, \hat{e}_\theta + i\hat{e}_\phi, \hat{e}_\theta - i\hat{e}_\phi\}$ is linearly independent, we obtain the following ordinary differential equations:

$$\begin{aligned} &\frac{\partial}{\partial \xi} \frac{1}{\xi^2} \frac{\partial}{\partial \xi} (\xi^2 f) - \frac{l(l+1)}{\xi^2} f - \frac{1}{\xi^2} g_1 + \frac{1}{\xi^2} g_2 + \xi^2 \frac{\partial^2}{\partial \xi^2} f + 2\xi\eta \frac{\partial^2}{\partial \xi \partial \eta} f \\ &\quad - (1 - \eta^2) \frac{\partial^2}{\partial \eta^2} f + 2\xi \frac{\partial}{\partial \xi} f + 2\eta \frac{\partial}{\partial \eta} f - \alpha f = 0 \end{aligned} \tag{2.6}$$

$$\begin{aligned} &\frac{1}{2\xi} \frac{\partial^2}{\partial \xi^2} (\xi g_2) - \frac{l(l+1)}{2\xi^2} g_2 + \frac{l(l+1)}{\xi^2} f + \frac{1}{2} \xi^2 \frac{\partial^2}{\partial \xi^2} g_2 + \xi\eta \frac{\partial^2}{\partial \xi \partial \eta} g_2 \\ &\quad - \frac{1}{2} (1 - \eta^2) \frac{\partial^2}{\partial \eta^2} g_2 + \xi \frac{\partial}{\partial \xi} g_2 + \eta \frac{\partial}{\partial \eta} g_2 - \frac{\alpha}{2} g_2 = 0 \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\frac{1}{2\xi} \frac{\partial^2}{\partial \xi^2} (\xi g_1) - \frac{l(l+1)}{2\xi^2} g_1 - \frac{l(l+1)}{\xi^2} f + \frac{1}{2} \xi^2 \frac{\partial^2}{\partial \xi^2} g_1 + \xi\eta \frac{\partial^2}{\partial \xi \partial \eta} g_1 \\ &\quad - \frac{1}{2} (1 - \eta^2) \frac{\partial^2}{\partial \eta^2} g_1 + \xi \frac{\partial}{\partial \xi} g_1 + \eta \frac{\partial}{\partial \eta} g_1 - \frac{\alpha}{2} g_1 = 0 \end{aligned} \tag{2.8}$$

Then, introducing the functions $G \equiv (g_1 + g_2)/2$ and $H \equiv (g_2 - g_1)/2$ and using equations (2.7) and (2.8), we obtain the following partial differential equation:

$$(1 + \xi^2)\frac{\partial^2 G}{\partial \xi^2} + \frac{2}{\xi}(1 + \xi^2)\frac{\partial G}{\partial \xi} + 2\xi\eta\frac{\partial^2 G}{\partial \xi \partial \eta}$$

$$- (1 - \eta^2)\frac{\partial^2 G}{\partial \eta^2} + 2\eta\frac{\partial G}{\partial \eta} - \left[\frac{\mu}{\xi^2} + \alpha\right] G = 0 \qquad (2.9)$$

where $\mu = l(l + 1)$.

To solve this differential equation, we introduce the change of independent variables defined by

$$\xi = \rho \cosh \tau \qquad \text{and} \qquad \eta = \rho \sinh \tau \qquad (2.10)$$

and we obtain the following partial differential equation:

$$\left\{(1 + \rho^2)\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}(3 + 2\rho^2)\frac{\partial}{\partial \rho}\right.$$

$$\left. - \frac{1}{\rho^2}\frac{\partial^2}{\partial \tau^2} - 2\frac{\tanh\tau}{\rho^2}\frac{\partial}{\partial \tau} - \frac{M_0^2}{1 + \rho^2} - \frac{l(l + 1)}{\rho^2\cosh^2\tau}\right\} G(\rho, \tau) = 0 \qquad (2.11)$$

Then, we suppose a separable solution as follows:

$$G(\rho, \tau) = F(\rho)T_l(\tau) \qquad (2.12)$$

in the above equation and we obtain two ordinary differential equations given by

$$\rho^2(1 + \rho^2)F'' + \rho(3 + 2\rho^2)F' - \frac{M_0^2\rho^2}{1 + \rho^2}F - \lambda_0^2 F = 0 \qquad (2.13)$$

and

$$T_l'' + 2\tanh\tau\, T_l' + \frac{l(l + 1)}{\cosh^2\tau}T_l - \lambda_0^2 T_l = 0 \qquad (2.14)$$

where λ_0^2 is a constant and the prime denotes differentiation.

Taking $\lambda_0^2 = n(n + 2)$, where $n = 0, 1, 2, \ldots,$ we obtain the solution of equation (2.14) in the r and t variables as

$$T_l(r, t) = \left(1 - \frac{c^2t^2}{r^2}\right)^{(n+2)/2} C_{l-n+1}^{n+3/2}\left(\frac{ct}{r}\right) \qquad (2.15)$$

where $C_\mu^\nu(x)$ are the Gegenbauer polynomials.

The solution of equation (2.13) is given by

$$F(\rho) = \exp\left[-\frac{\pi}{2}(n + \nu + 3)\right]\left(\frac{A_0^2}{A_0^2 - 1}\right)^{3/4} \frac{\Gamma(n - \nu)}{\Gamma(n + \nu + 3)} 2^{\nu+3/2}\Gamma(\nu + 3/2)$$
$$\times \left\{P_{n+1/2}^{\nu+3/2}\left(\frac{1}{\sqrt{1 - A_0^2}}\right) + \frac{2i}{\pi} Q_{n+1/2}^{\nu+3/2}\left(\frac{1}{\sqrt{1 - A_0^2}}\right)\right\} \tag{2.16}$$

where $M_0^2 = \nu(\nu + 3)$, $A_0^2 = 1 + \xi^2 - \eta^2 \equiv 1 + \rho^2$, and $P_n^\nu(x)$ and $Q_n^\nu(x)$ are the Legendre associated functions of the first and second kinds, respectively.

Then, rewriting equations (2.6) and using equations (2.7) and (2.8), we obtain

$$(1 + \xi^2)\frac{\partial^2 f}{\partial\xi^2} + \frac{2}{\xi}(1 + \xi^2)\frac{\partial f}{\partial\xi} - \frac{l(l + 1)}{\xi^2} f + \frac{2}{\xi^2} H - \frac{2}{\xi^2} f + 2\xi\eta\frac{\partial^2 f}{\partial\xi\partial\eta}$$
$$- (1 - \eta^2)\frac{\partial^2 f}{\partial\eta^2} + 2\eta\frac{\partial f}{\partial\eta} - \alpha f = 0 \tag{2.17}$$

and

$$(1 + \xi^2)\frac{\partial^2 H}{\partial\xi^2} + \frac{2}{\xi}(1 + \xi^2)\frac{\partial H}{\partial\xi} - \frac{l(l + 1)}{\xi^2} H + \frac{2l(l + 1)}{\xi^2} f + 2\xi\eta\frac{\partial^2 H}{\partial\xi\partial\eta}$$
$$-(1 - \eta^2)\frac{\partial^2 H}{\partial\eta^2} + 2\eta\frac{\partial H}{\partial\eta} - \alpha H = 0 \tag{2.18}$$

Multiplying equation (2.17) by an arbitrary constant k and using equation (2.18), we obtain

$$(1 + \xi^2)\frac{\partial^2}{\partial\xi^2}(kf + H) + \frac{2}{\xi}(1 + \xi^2)\frac{\partial}{\partial\xi}(kf + H) - \frac{l(l + 1)}{\xi^2}(kf + H)$$
$$+ \left[\frac{2l(l + 1)}{k} - 2\right]\frac{kf}{\xi^2} + \frac{2k}{\xi^2} H + 2\xi\eta\frac{\partial^2}{\partial\xi\partial\eta}(kf + H)$$
$$- (1 - \eta^2)\frac{\partial^2}{\partial\eta^2}(kf + H) + 2\eta\frac{\partial}{\partial\eta}(kf + H) - \alpha(kf + H) = 0 \tag{2.19}$$

Then, choosing the constant k such that

$$\frac{2l(l + 1)}{k} - 2 = 2k \tag{2.20}$$

we obtain $k = l$ and $k = -l - 1$. First, for $k = l$, equation (2.19) can be written as

$$\left\{(1+\xi^2)\frac{\partial^2}{\partial\xi^2}+\frac{2}{\xi}(1+\xi^2)\frac{\partial}{\partial\xi}+2\xi\eta\frac{\partial^2}{\partial\xi\partial\eta}\right.$$
$$\left.-(1-\eta^2)\frac{\partial^2}{\partial\eta^2}+2\eta\frac{\partial}{\partial\eta}-\left[\frac{l(l-1)}{\xi^2}+\alpha\right]\right\}(lf+H)=0 \quad (2.21)$$

and the solution of this differential equation is given by equation (2.9) making $l \to -l$ as follows:

$$(lf+H)(\rho,\tau)=F(\rho)T_{-l}(\tau) \quad (2.22)$$

Next, for $k = -l - 1$, we obtain from equation (2.19)

$$\left\{(1+\xi^2)\frac{\partial^2}{\partial\xi^2}+\frac{2}{\xi}(1+\xi^2)\frac{\partial}{\partial\xi}+2\xi\eta\frac{\partial^2}{\partial\xi\partial\eta}\right.$$
$$\left.-(1-\eta^2)\frac{\partial^2}{\partial\eta^2}+2\eta\frac{\partial}{\partial\eta}-\left[\frac{(l+1)(l+2)}{\xi^2}+\alpha\right]\right\}[H-(l+1)f]=0$$
$$(2.23)$$

and the solution of this differential equation is given by equation (2.9) making the substitution $l \to l+1$

$$[H-(l+1)f](\rho,\tau)=F(\rho)T_{l+1}(\tau) \quad (2.24)$$

where the functions $F(\rho)$ and $T_{-l}(\tau)[T_{l+1}(\tau)]$ are given by equations (2.16) and (2.15), respectively.

Solving the system of equations (2.12), (2.22), and (2.24) for the functions $g_1(\xi,\eta)$, $g_2(\xi,\eta)$, and $f(\xi,\eta)$ and introducing these functions in equations (2.5), we can write for $l > 0$

$$\Psi_\xi=\frac{\sqrt{l(l+1)}}{2l+1}F(\rho)[T_{-l}(\tau)-T_{l+1}(\tau)]\,Y_{lm}(\theta,\phi)$$

$$\Psi_+=\frac{l}{2l+1}F(\rho)\left[\frac{2l+1}{l}T_l(\tau)-T_{l+1}(\tau)-\frac{l+1}{l}T_{-l}(\tau)\right]{}_1Y_{lm}(\theta,\phi)$$

$$\Psi_-=\frac{l}{2l+1}F(\rho)\left[T_{l+1}(\tau)+\frac{2l+1}{l}T_l(\tau)+\frac{l+1}{l}T_{-l}(\tau)\right]{}_{-1}Y_{lm}(\theta,\phi)$$
$$(2.25)$$

As a particular case we consider $l = 0$. Then the components $\Psi_\pm$ are equal to zero [7] and we assume for equation (2.4) a solution of the form

$$\Psi_\xi=f(\xi,\eta);\qquad \Psi_+=0;\qquad \Psi_-=0 \quad (2.26)$$

We note that $l = 0$ implies $m = 0$ and thus $Y_{00}(\theta, \phi)$ is a constant.

Introducing the functions (2.16) in equation (2.4), we get

$$\left[(1 + \xi^2)\frac{\partial^2}{\partial\xi^2} + \frac{2}{\xi}(1 + \xi^2)\frac{\partial}{\partial\xi} + 2\xi\eta\frac{\partial^2}{\partial\xi\partial\eta} - (1 - \eta^2)\frac{\partial^2}{\partial\eta^2} + 2\eta\frac{\partial}{\partial\eta} - \left(\frac{2}{\xi} + \alpha\right)\right] f(\xi, \eta) = 0 \tag{2.27}$$

The solution of the above equation is given by equation (2.9) with $l = 1$

$$f(\rho, \tau) = F(\rho)T_1(\tau)$$

where $F(\rho)$ and $T_1(\tau)$ are given by equations (2.16) and (2.15), respectively and the variables ρ and τ are defined in equation (2.10). Then, the vector solution for equation (2.4) with $l = 1$ is given by

$$\Psi = F(\rho)T_1(\tau)\cdot\hat{e}_\rho$$

where $\Psi \equiv \Psi(\rho, \tau, \theta, \phi)$.

3. DIRAC WAVE EQUATION

In this section we present and discuss the Dirac wave equation in the de Sitter universe with spherical coordinates and we obtain the solution for the stationary case using the spin-weight technique.

The Dirac wave equation is given by [5]

$$\left\{\frac{1}{2}\gamma_a\,\gamma_b\,L_{ab} - Rm\right\}\Psi = 0 \tag{3.1}$$

where the γ_c are 4×4 matrices that satisfy

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\delta_{ab}$$

and the L_{ab} are the angular momentum operators, defined by

$$L_{ab} = -i\hbar\left(\xi_a\frac{\partial}{\partial\xi_b} - \xi_b\frac{\partial}{\partial\xi_a}\right)$$

with $a, b = 0, 1, \ldots, 4$; m is a real scalar and R is the radius of the de Sitter universe.

We obtain the explicit form for equation (3.1) in relativistic spherical coordinates using the technique proposed by Fantappié and Arcidiacono as follows [5]:

$$A\frac{\partial\Psi}{\partial t} + B\frac{\partial\Psi}{\partial r} + C\frac{\partial\Psi}{\partial\theta} + D\frac{\partial\Psi}{\partial\phi} - \frac{Rm}{i\hbar}I_4\Psi = 0 \tag{3.2}$$

where

$$A = \begin{bmatrix} R(1 + t^2/R^2)I_2 & -r(1 - t/R)\sigma_r \\ -r(1 + t/R)\sigma_r & -R(1 + t^2/R^2)I_2 \end{bmatrix}$$

$$B = \begin{bmatrix} (rt/R)I_2 & \{t + (r^2 + R^2)/R\}\sigma_r \\ \{t - (r^2 + R^2)/R\}\sigma_r & -rt/RI_2 \end{bmatrix}$$

$$C = \begin{bmatrix} -i\sigma_\phi & \{(t + R)/r\}\sigma_\theta \\ \{(t - R)/r\}\sigma_\theta & -i\sigma_\phi \end{bmatrix}$$

and

$$D = \begin{bmatrix} ir\sigma_\theta & (t + R)\sigma_\phi \\ (t - R)\sigma_\phi & ir\sigma_\theta \end{bmatrix}\frac{1}{r\sin\theta}$$

with $\sigma_\theta \equiv \sigma \cdot \hat{e}_\theta$; $\sigma_\phi \equiv \sigma \cdot \hat{e}_\phi$; $\sigma_r \equiv \sigma \cdot \hat{e}_r$. Here I_2 and I_4 are respectively the 2×2 and 4×4 identity matrices, and we use for the matrices γ_a the following representation:

$$\gamma_\lambda = \gamma_0\alpha_\lambda \qquad \text{and} \qquad \gamma_4 = \gamma_0\gamma_1\gamma_2\gamma_3$$

where

$$\alpha_\lambda = \begin{bmatrix} 0 & \sigma_\lambda \\ \sigma_\lambda & 0 \end{bmatrix}, \qquad \gamma_0 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}$$

and α_λ are the Pauli matrices, with $\lambda = 1, 2, 3$.

If we put $\Psi = [{}^u_v]$, where u and v are two-component spinors, and then write $u = u_-\vartheta - u_+l$ and $v = v_-\vartheta - v_+l$, where the spinors ϑ and l were introduced in [6, 7] we can write the following identities:

$$\sigma \cdot \nabla u = \sigma_r\frac{\partial u}{\partial r} + \sigma_\phi\frac{1}{r\sin\theta}\frac{\partial u}{\partial\phi} + \sigma_\theta\frac{1}{r}\frac{\partial u}{\partial\theta}$$

$$\sigma \cdot \nabla(u_-\vartheta) = \left[\frac{1}{r}\frac{\partial}{\partial r}(ru_-)\right]\vartheta + \left(\frac{1}{r}\partial\, u_-\right)l \tag{3.3}$$

$$\sigma \cdot \nabla(u_+ \, l) = \left(\frac{1}{r}\bar{\partial}\, u_+\right)\vartheta - \left[\frac{1}{r}\frac{\partial}{\partial r}(ru_+)\right]l$$

and we also note that the following identities hold:

$$\frac{\partial\vartheta}{\partial\phi} = \frac{i}{2}\,\delta\vartheta; \qquad \frac{\partial\vartheta}{\partial\theta} = -\frac{1}{2}\,l; \qquad \frac{\partial l}{\partial\theta} = \frac{1}{2}\,\vartheta; \qquad \frac{\partial l}{\partial\phi} = \frac{i}{2}\,\delta l$$

$$\sigma_\theta\vartheta = -l; \qquad \sigma_\phi\vartheta = -il; \qquad \sigma_r\vartheta = \vartheta; \qquad \sigma_\theta l = -\vartheta; \qquad \sigma_\phi l = i\vartheta$$

$$\sigma_r l = -l; \qquad \sigma_\theta\delta\vartheta = \sin\theta\,\vartheta + \cos\theta\, l; \qquad \sigma_\theta\delta l = \sin\theta\, l - \cos\theta\,\vartheta \tag{3.4}$$

where $\delta \equiv \left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$.

Then, using identities (3.3) and (3.4) and the fact that the set $\{\vartheta, -l\}$ is linearly independent, we can write equation (3.2) as a system of four partial differential equations:

$$\begin{aligned}
&R\left(1 + \frac{t^2}{R^2}\right)\frac{\partial u_-}{\partial t} - \left(1 - \frac{t}{R}\right)\frac{\partial v_-}{\partial t} + (t + R)\frac{\partial v_-}{\partial r} - (t + R)\frac{1}{r}\bar{\partial} v_+ \\
&\quad + \frac{r^2}{R}\frac{\partial v_-}{\partial r} + \frac{rt}{R}\frac{\partial u_-}{\partial r} + \bar{\partial} u_+ - (1 + k)u_- + (t + R)\frac{1}{r}\, v_- = 0 \\
&R\left(1 + \frac{t^2}{R^2}\right)\frac{\partial v_+}{\partial r} + r\left(1 - \frac{t}{R}\right)\frac{\partial v_+}{\partial t} - (t + R)\frac{\partial v_+}{\partial r} - (t + R)\frac{1}{r}\,\partial v_- \\
&\quad - \frac{r^2}{R}\frac{\partial v_+}{\partial r} + \frac{rt}{R}\frac{\partial u_+}{\partial r} - \partial u_- - (1 + k)u_+ - (t + R)\frac{1}{r}\, v_+ = 0 \\
&r\left(1 + \frac{t}{R}\right)\frac{\partial u_-}{\partial t} + R\left(1 + \frac{t^2}{R^2}\right)\frac{\partial v_-}{\partial t} - (t - R)\frac{\partial u_-}{\partial r} + (t - R)\frac{1}{r}\bar{\partial} u_+ \\
&\quad + \frac{r^2}{R}\frac{\partial u_-}{\partial r} + \frac{rt}{R}\frac{\partial v_-}{\partial r} - \bar{\partial} v_+ + (1 + k)v_- - (t - R)\frac{1}{r}\, u_- = 0 \\
&r\left(1 + \frac{t}{R}\right)\frac{\partial u_+}{\partial t} - R\left(1 + \frac{t^2}{R^2}\right)\frac{\partial v_+}{\partial t} - (t - R)\frac{\partial u_+}{\partial r} - (t - R)\frac{1}{r}\,\partial u_- \\
&\quad + \frac{r^2}{R}\frac{\partial u_+}{\partial r} - \frac{rt}{R}\frac{\partial v_+}{\partial r} - \partial v_- - (1 + k)v_+ - (t - R)\frac{1}{r}\, u_+ = 0
\end{aligned} \tag{3.5}$$

where $k \equiv Rm/i\hbar$.

Equations (3.5) can be solved using the method of separation of variables. Using the fact that the set of spherical harmonics with spin weight is complete, we look for a solution in the following form:

$$\begin{aligned} u_- &\equiv g(r,t)\ {}_{-1/2}Y_{jm}(\theta,\phi) \\ u_+ &\equiv G(r,t)\ {}_{1/2}Y_{jm}(\theta,\phi) \\ v_- &\equiv f(r,t)\ {}_{-1/2}Y_{jm}(\theta,\phi) \\ v_+ &\equiv F(r,t)\ {}_{1/2}Y_{jm}(\theta,\phi) \end{aligned} \tag{3.6}$$

where $j \geq 1/2$, $-j \leq m \leq j$, and we have used the fact that the components u_- and v_- have spin weight $-1/2$ and the components u_+ and v_+ have spin weight 1/2.

Introducing the functions (3.6) in equations (3.5) and using the relations [7]

$$\eth_{-1/2}Y_{jm}(\theta,\phi) = \left(j + \frac{1}{2}\right) {}_{-1/2}Y_{jm}(\theta,\phi)$$

$$\eth_{1/2}Y_{jm}(\theta,\phi) = -\left(j + \frac{1}{2}\right) {}_{-1/2}Y_{jm}(\theta,\phi)$$

we obtain the following system of partial differential equations:

$$R\left(1 + \frac{t^2}{R^2}\right)\frac{\partial g}{\partial t} - \left(1 - \frac{t}{R}\right)\frac{\partial f}{\partial t} + \left(t + R + \frac{r^2}{R}\right)\frac{\partial f}{\partial r} + \frac{rt}{R}\frac{\partial g}{\partial r}$$
$$= \left(j + \frac{1}{2}\right) G - (t + R)\left(j + \frac{1}{2}\right)\frac{1}{r} F + (1 + k)g - (t + R)\frac{1}{r} f$$

$$R\left(1 + \frac{t^2}{R^2}\right)\frac{\partial G}{\partial t} + r\left(1 - \frac{t}{R}\right)\frac{\partial F}{\partial t} - \left(t + R + \frac{r^2}{R}\right)\frac{\partial F}{\partial r} + \frac{rt}{R}\frac{\partial G}{\partial r}$$
$$= \left(j + \frac{1}{2}\right) g + (t + R)\left(j + \frac{1}{2}\right)\frac{1}{r} f + (1 + k)G + (t + R)\frac{1}{r} F$$

$$r\left(1 + \frac{t}{R}\right)\frac{\partial g}{\partial t} + R\left(1 + \frac{t^2}{R^2}\right)\frac{\partial f}{\partial t} - \left(t - R - \frac{r^2}{R}\right)\frac{\partial g}{\partial r} + \frac{rt}{R}\frac{\partial f}{\partial r}$$
$$= (t - R)\left(j + \frac{1}{2}\right)\frac{1}{r} G - \left(j + \frac{1}{2}\right) F + (t - R)\frac{1}{r} g - (1 + k)f$$

$$r\left(1+\frac{t}{R}\right)\frac{\partial G}{\partial t}-R\left(1+\frac{t^2}{R^2}\right)\frac{\partial F}{\partial t}-\left(t-R-\frac{r^2}{R}\right)\frac{\partial G}{\partial r}-\frac{rt}{R}\frac{\partial F}{\partial r}$$
$$=(t-R)\left(j+\frac{1}{2}\right)\frac{1}{r}g+\left(j+\frac{1}{2}\right)f+(t-R)\frac{1}{r}G+(1+k)F \qquad (3.7)$$

We note that solving the above system implies finding a completely explicit solution for the Dirac wave question in the de Sitter universe.

We now solve the above system for the stationary case, i.e., when the functions g, G, f, and F are independent of time. In this case, we have

$$\begin{cases}(1+\rho^2)\dfrac{dA}{d\rho}+\dfrac{1}{\rho}\left(j+\dfrac{3}{2}\right)A=\left(k+\dfrac{1}{2}-j\right)B\\[2mm](1+\rho^2)\dfrac{dB}{d\rho}+\dfrac{1}{\rho}\left(\dfrac{1}{2}-j\right)B=-\left(k+\dfrac{3}{2}+j\right)A\end{cases} \qquad (3.8)$$

$$\begin{cases}(1+\rho^2)\dfrac{dD}{d\rho}+\dfrac{1}{\rho}\left(j+\dfrac{3}{2}\right)D=-\left(k+\dfrac{1}{2}-j\right)C\\[2mm](1+\rho^2)\dfrac{dC}{d\rho}+\dfrac{1}{\rho}\left(\dfrac{1}{2}-j\right)C=\left(k+\dfrac{3}{2}+j\right)D\end{cases} \qquad (3.9)$$

where we have put

$$A\equiv g+G; \qquad B\equiv F-f; \qquad C\equiv G-g; \qquad D\equiv F+f$$

with $\rho = r/R$.

Then, from the system (3.8) we obtain the following ordinary differential equation:

$$\frac{d^2A}{d\rho^2}+\frac{2}{\rho}\frac{dA}{d\rho}-\frac{(j+1/2)(j+3/2)}{(1+\rho^2)\rho^2}A+\frac{k(k+2)}{(1+\rho^2)^2}A=0 \qquad (3.10)$$

The solution of the above equation is given by [8]

$$A(\rho)=\frac{1}{\rho}(1+\rho^2)^{-k/2}\,F_k^{j+1/2}(\rho)\,\hat{F}_k^{j+1/2}(\rho) \qquad (3.11)$$

where $F_k^l(\rho)$ and $\hat{F}_k^l$ are given by

$$F_k^l(\rho)=\rho^{l+1}\sum_{n=0}^{[(k-l)/2]}(-1)^n\frac{(k-l+1)!\,\Gamma(l+3/2)}{(k-l-2n)!\,\Gamma(n+l+3/2)}\frac{1}{n!}\left(\frac{\rho}{2}\right)^{2n}$$

and

$$\hat{F}_k^l(\rho) = \frac{1}{\rho^l} \sum_{n=0}^{[(k+l+1)/2]} (-1)^n \frac{(k+l+1)!\, \Gamma(-l+1/2)}{(k+l+1-2n)!\, \Gamma(n-l+1/2)} \frac{1}{n!} \left(\frac{\rho}{2}\right)^{2n}$$

Introducing the function (3.11) in the system (3.8), we obtain

$$\begin{aligned}
&\left(k+\frac{1}{2}-j\right)\Bigg)\rho(1+\rho^2)^{k/2} B(\rho) \\
&\quad = \hat{F}_k^{j+1/2}(\rho)\left(\left\{\frac{1+\rho^2}{\rho}\left[\frac{4(j+3/2)(j+7/2)}{(2j+6)} + \frac{4(j+1/2)(j+5/2)}{(2j+4)} - 1\right]\right.\right. \\
&\qquad \left. + (2k-2j-2)\rho - \frac{k}{2}\right\} F_k^{j+3/2}(\rho) \\
&\qquad - \frac{1}{\rho}\left[\frac{(k-j+1/2)(k+j+5/2)}{(2j+4)} + \frac{(k+j+7/2)(k-j-3/2)}{(2j+6)}\right] F_k^{j+1/2}(\rho) \\
&\qquad \left. + (j+3/2)\frac{1}{\rho} F_k^{j+1/2}(\rho)\right)
\end{aligned} \tag{3.12}$$

where we have used the following relations for the $F_k^l(\rho)$ and $\hat{F}_k^l(\rho)$ polynomials [9]:

$$\begin{aligned}
&\rho(1+\rho^2)(2l+3)\frac{d}{d\rho} F_k^l(\rho) \\
&\quad = [(k-l)(2l+3)\rho^2 + 4l(l+2)(1+\rho^2)]\, F_k^l(\rho) \\
&\quad - (k-l+1)(k+l+2) F_k^{l+1}(\rho)
\end{aligned}$$

and

$$\begin{aligned}
&\rho(1+\rho^2)(2l+5)\frac{d}{d\rho}\hat{F}_k^l(\rho) \\
&\quad = -(k+l+3)(k-l-1)\,\hat{F}_k^{l+1}(\rho) \\
&\qquad + [k-l-1)(2l+5)\rho^2 + 4(l+1)(l+3)(1+\rho^2)]\hat{F}_k^l(\rho)
\end{aligned}$$

Analogously, we solve the system (3.9) and we obtain the solution

$$C(\rho) = -B(\rho) \tag{3.13}$$

$$D(\rho) = A(\rho) \tag{3.14}$$

Finally, the solution of the Dirac wave equation (stationary case) in the de Sitter universe is given by

$$\begin{bmatrix} u_- \\ u_+ \\ v_- \\ v_+ \end{bmatrix} = \begin{bmatrix} A(\rho) X^m_{j+1/2} \\ B(\rho) X^m_{-j-1/2} \end{bmatrix} + \begin{bmatrix} C(\rho) X^m_{-j-1/2} \\ D(\rho) X^m_{j+1/2} \end{bmatrix} \tag{3.15}$$

where the functions A, B, C, and D are given by equations (3.11), (3.12), (3.13), and (3.14) respectively and the functions $X^m_{\pm j \pm 1/2}$ are given by

$$X^m_{j+1/2} = \frac{1}{2} \begin{bmatrix} {}_{-1/2}Y_{jm} \\ {}_{1/2}Y_{jm} \end{bmatrix} \quad \text{and} \quad X^m_{-j-1/2} = \frac{1}{2} \begin{bmatrix} -{}_{-1/2}Y_{jm} \\ {}_{1/2}Y_{jm} \end{bmatrix} \tag{3.16}$$

4. COMMENTS

In this paper we discussed the solution for the Klein–Gordon and Dirac wave equations using the so-called Fantappié–Arcidiacono method and Newman–Penrose spin-weight technique. In both cases the solutions are given by means of the spinor spherical harmonics. For the radial Dirac wave equation the solution is given by a new class of polynomials which is related to the so-called relativistic Hermite polynomials recently introduced [10].

We also note that as the curvature of de Sitter spacetime goes to zero ($R \to \infty$) we obtain the classical results [5].

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